

# LEBESGUE-TYPE INEQUALITIES FOR QUASI-GREEDY BASES

EUGENIO HERNÁNDEZ

**ABSTRACT.** We show that for quasi-greedy bases in real Banach spaces the error of the thresholding greedy algorithm of order  $N$  is bounded by the best  $N$ -term error of approximation times a constant which depends on the democracy functions and the quasi-greedy constant of the basis.

## 1. INTRODUCTION

Let  $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$  be a real Banach space with a countable seminormalized basis  $\mathcal{B} = \{e_k : k \in \mathbb{N}\}$ . Let  $\Sigma_N, N = 1, 2, 3, \dots$  be the set of all  $y \in \mathbb{B}$  with at most  $N$  non-null coefficients in the unique basis representation. For  $x \in \mathbb{B}$ , the  **$N$ -term error of approximation** with respect to  $\mathcal{B}$  is

$$\sigma_N(x) = \sigma_N(x; \mathcal{B}, \mathbb{B}) := \inf_{y \in \Sigma_N} \|x - y\|_{\mathbb{B}}, \quad N = 1, 2, 3, \dots$$

Given  $x = \sum_{k \in \mathbb{N}} a_k(x) e_k \in \mathbb{B}$ , let  $\pi$  denote any bijection of  $\mathbb{N}$  such that

$$|a_{\pi(k)}(x)| \geq |a_{\pi(k+1)}(x)| \quad \text{for all } k \in \mathbb{N}. \quad (1.1)$$

The **thresholding greedy algorithm of order  $N$**  (TGA) is defined by

$$G_N(x) = G_N^{\pi}(x; \mathcal{B}, \mathbb{B}) := \sum_{k=1}^N a_{\pi(k)}(x) e_{\pi(k)}.$$

It is not always true that  $G_N(x) \rightarrow x$  (in  $\mathbb{B}$ ) as  $N \rightarrow \infty$ . A basis  $\mathcal{B}$  is called **quasi-greedy** if  $G_N(x) \rightarrow x$  (in  $\mathbb{B}$ ) as  $N \rightarrow \infty$  for all  $x \in \mathbb{B}$ . It turns out that this is equivalent (see Theorem 1 in [9]) to the existence of some constant  $C$  such that

$$\sup_N \|G_N(x)\|_{\mathbb{B}} \leq C \|x\|_{\mathbb{B}} \quad \text{for all } x \in \mathbb{B}. \quad (1.2)$$

It is convenient to define the **quasi-greedy constant  $K$**  to be the least constant such that

$$\|G_N(x)\|_{\mathbb{B}} \leq K \|x\|_{\mathbb{B}} \quad \text{and} \quad \|x - G_N(x)\|_{\mathbb{B}} \leq K \|x\|_{\mathbb{B}}, \quad x \in \mathbb{B}.$$

Given a basis  $\mathcal{B}$  in a Banach space  $\mathbb{B}$  a **Lebesgue-type inequality** is an inequality of the form

$$\|x - G_N(x)\|_{\mathbb{B}} \leq C v(N) \sigma_N(x), \quad x \in \mathbb{B},$$

where  $v(N)$  is a nondecreasing function of  $N$ . For a survey of Lebesgue-type inequalities see [6] and the references given there.

---

*Date:* November 17, 2011.

*2010 Mathematics Subject Classification.* 41A65, 41A46, 41A17.

*Key words and phrases.* Lebesgue-type inequalities, thresholding greedy algorithm, quasi-greedy bases, democracy functions.

Research supported by Grant MTM2010-16518 (Spain).

The purpose of this note is to find Lebesgue-type inequalities for quasi-greedy basis in a Banach space.

For a seminormalized collection  $\mathcal{B} = \{u_k\}_{k \in \mathbb{N}}$  in a Banach space  $\mathbb{B}$  the following quantities are defined:

$$h_r(N) = \sup_{|\Gamma|=N} \left\| \sum_{k \in \Gamma} u_k \right\|, \quad h_l(N) = \inf_{|\Gamma|=N} \left\| \sum_{k \in \Gamma} u_k \right\|,$$

and

$$\mu(N) = \sup_{1 \leq k \leq N} \frac{h_r(k)}{h_l(k)}, \quad N = 1, 2, 3, \dots \quad (1.3)$$

These functions are implicit in earlier works on  $N$ -term approximation and explicitly defined in [4]. The function  $\mu(N)$  is defined in [9]. The functions  $h_r$  and  $h_l$  are called right and left democracy functions of  $\mathcal{B}$  (see [2] and [3]).

Our main result is the following:

**Theorem 1.1.** *Let  $\mathcal{B} = \{e_k\}_{k=1}^\infty$  be a quasi-greedy basis in a real Banach space  $\mathbb{B}$ , and let  $K$  be the quasi-greedy constant of  $\mathcal{B}$ . Then for all  $N = 1, 2, 3, \dots$  and all  $x \in \mathbb{B}$ ,*

$$\|x - G_N(x)\|_{\mathbb{B}} \lesssim 8K^5 \left( \sum_{k=1}^N \mu(k) \frac{1}{k} \right) \sigma_N(x).$$

Before proving Theorem 1.1 we make some remarks about the function

$$v(N) := \sum_{k=1}^N \mu(k) \frac{1}{k}.$$

Obviously,  $\mu$  is increasing so that  $v(N) \lesssim \mu(N) \log N$ . In some cases this inequality is an equivalence. For example if  $\mu(N) \approx C$  (that is  $\mathcal{B}$  is democratic) then  $v(N) \approx \log N$ . In other cases the inequality can be improved. It can be proved that if  $\mathcal{B}$  is quasi-greedy,  $\mu$  is doubling, that is there exists a constant  $D \geq 1$  such that  $\mu(2k) \leq D\mu(k)$ ,  $k \in \mathbb{N}$  (see the Appendix). Under this condition it is not difficult to prove that

$$v(N) \approx \sum_{k=1}^{\log_2 N} \mu(2^k).$$

Moreover, if we assume that  $\mu$  has a positive dilation index, that is  $\mu \in \mathbb{W}_+$  in the terminology of [3], by Lemma 2.1 in [3] we have

$$v(N) \lesssim \mu(2^{\log_2 N}) = \mu(N),$$

so that in this situation we do not need the  $\log N$  factor.

We prove Theorem 1.1 in Section 2. Section 3 contains some comments and open questions.

## 2. PROOF OF THEOREM 1.1

We need the following result from [1]: let  $\mathcal{B} = \{e_k\}_{k=1}^\infty$  be a quasi-greedy bases with constant  $K$  in a real Banach space. For any finite set  $\Gamma \subset \mathbb{N}$  and any real numbers  $\{a_k\}_{k \in \Gamma}$  we have

$$\frac{1}{4K^2}(\min_{\Gamma} |a_k|) \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq \left\| \sum_{k \in \Gamma} a_k e_k \right\|_{\mathbb{B}} \leq (2K)(\max_{\Gamma} |a_k|) \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \quad (2.1)$$

(see Lemma 2.1 and Lemma 2.2 in [1]).

**Lemma 2.1.** *Let  $\mathcal{B}$  and  $\mathbb{B}$  as in Theorem 1.1. Suppose that there exists  $C_1 > 0$  such that for all  $\Gamma \subset \mathbb{N}$  finite*

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \leq C_1 \eta(|\Gamma|) \quad (2.2)$$

for some  $\eta$  increasing and doubling (for example,  $\eta(N) = h_r(N)$  and  $C_1 = 1$ ). Then there exists  $C = C_\eta$  such that for any  $x = \sum_{k \in \mathbb{N}} a_k(x) e_k \in \mathbb{B}$

$$\|x\|_{\mathbb{B}} \leq 2K C_\eta \sum_{k=1}^{\infty} a_k^*(x) \eta(k) \frac{1}{k}, \quad (2.3)$$

where  $\{a_k^*(x)\}$  is a decreasing rearrangement of  $\{|a_k(x)|\}$  as in (1.1).

*Proof.* Let  $\pi$  be a bijection of  $\mathbb{N}$  that gives  $\{a_k^*(x)\}$ , that is  $\{a_k^*(x)\} = |a_{\pi_k}(x)|$ . Since  $\mathcal{B}$  is quasi-greedy

$$\lim_{N \rightarrow \infty} \sum_{k=1}^N a_{\pi_k}(x) e_{\pi_k} \rightarrow x \text{ (convergence in } \mathbb{B} \text{)}.$$

Thus

$$\begin{aligned} \|x\|_{\mathbb{B}} &= \left\| \sum_{k=1}^{\infty} a_{\pi(k)}(x) e_{\pi(k)} \right\|_{\mathbb{B}} = \left\| \sum_{j=0}^{\infty} \sum_{2^j \leq k < 2^{j+1}} a_{\pi(k)}(x) e_{\pi(k)} \right\|_{\mathbb{B}} \\ &\leq \sum_{j=0}^{\infty} \left\| \sum_{2^j \leq k < 2^{j+1}} a_{\pi(k)}(x) e_{\pi(k)} \right\|_{\mathbb{B}}. \end{aligned}$$

We now use first the right hand side inequality of (2.1) and then condition (2.2) to deduce

$$\begin{aligned} \|x\|_{\mathbb{B}} &= \sum_{j=0}^{\infty} (2K) |a_{\pi(2^j)}(x)| \left\| \sum_{2^j \leq k < 2^{j+1}} e_{\pi(k)} \right\|_{\mathbb{B}} \leq (2K) C_1 \sum_{j=0}^{\infty} |a_{\pi(2^j)}(x)| \eta(2^j) \\ &= (2K) C_1 \sum_{j=0}^{\infty} a_{2^j}^*(x) \eta(2^j). \end{aligned}$$

Inequality (2.3) follows since  $\eta$  is doubling and increasing.  $\square$

**Lemma 2.2.** *Let  $\mathcal{B}$  and  $\mathbb{B}$  as in Theorem 1.1. Suppose that there exists  $C_2 > 0$  such that for all  $\Gamma \subset \mathbb{N}$  finite*

$$\frac{1}{C_2} \eta(|\Gamma|) \leq \left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{B}} \quad (2.4)$$

for some function  $\eta$  (for example,  $\eta(N) = h_l(N)$  and  $C_2 = 1$ ). Then for any  $x = \sum_{k \in \mathbb{N}} a_k(x) e_k \in \mathbb{B}$

$$\left[ \sup_{\Gamma} a_k^*(x) \eta(k) \right] \leq C_2 (4K^3) \|x\|_{\mathbb{B}}, \quad (2.5)$$

where  $\{a_k^*(x)\}$  is a decreasing rearrangement of  $\{|a_k(x)|\}$  as in (1.1).

*Proof.* Let  $\pi$  be as in the proof of Lemma 2.1. For any  $k \in \mathbb{N}$  we use condition (2.4) and then the left hand side inequality of (2.1) to obtain

$$|a_{\pi(k)}(x)|\eta(k) \leq C_2 |a_{\pi(k)}(x)| \left\| \sum_{j=1}^k e_{\pi(j)} \right\| \leq C_2 (4K^2) \left\| \sum_{j=1}^k a_{\pi(j)} e_{\pi(j)} \right\|.$$

We use (1.2) to deduce  $|a_{\pi(k)}(x)|\eta(k) \leq C_2 (4K^3) \|x\|_{\mathbb{B}}$ . The result follows by taking the supremum on  $k \in \Gamma$ .  $\square$

For  $\Gamma \subset \mathbb{N}$  and  $x = \sum_{k \in \mathbb{N}} a_k(x) e_k \in \mathbb{B}$  define the **projection operator over  $\Gamma$**  as

$$S_{\Gamma}(x) := \sum_{k \in \Gamma} a_k(x) e_k.$$

**Lemma 2.3.** *Let  $\mathcal{B}$  and  $\mathbb{B}$  as in Theorem 1.1. For  $\Gamma \subset \mathbb{N}$  finite*

$$\|S_{\Gamma}(x)\|_{\mathbb{B}} \lesssim (8K^4) \left( \sum_{k=1}^{|\Gamma|} \mu(k) \frac{1}{k} \right) \|x\|_{\mathbb{B}}.$$

*Proof.* Apply Lemma 2.1 with  $\eta(N) = h_r(N)$  to obtain

$$\begin{aligned} \|S_{\Gamma}(x)\|_{\mathbb{B}} &= \left\| \sum_{k \in \Gamma} a_k(x) e_k \right\|_{\mathbb{B}} \lesssim (2K) \sum_{k=1}^{|\Gamma|} a_k^*(x) h_r(k) \frac{1}{k} \\ &\leq (2K) \sum_{k=1}^{|\Gamma|} a_k^*(x) \frac{h_r(k)}{h_l(k)} h_l(k) \frac{1}{k} \\ &\leq (2K) [\sup_k a_k^*(x) h_l(k)] \sum_{k=1}^{|\Gamma|} \mu(k) \frac{1}{k}. \end{aligned}$$

Use now Lemma (2.2) with  $\eta(k) = h_l(k)$  to deduce the result.  $\square$

We now prove Theorem 1.1. The proof follows arguments used in [4], [7] and [8] that were presented by V. N. Temlyakov at the *Concentration week on greedy algorithms in Banach spaces and compressed sensing* held on July 18-22 at Texas A&M University.

Take  $\epsilon > 0$  and  $N = 1, 2, 3, \dots$ . Choose  $p_N(x) = \sum_{k \in P} b_k e_k$  with  $|P| = N$  such that

$$\|x - p_N(x)\|_{\mathbb{B}} \leq \sigma_N(x) + \epsilon. \quad (2.6)$$

Let  $\Gamma$  be the set of indices picked by the thresholding greedy algorithm after  $N$  iterations, that is

$$G_N(x) = \sum_{k \in \Gamma} a_k(x) e_k, \quad |\Gamma| = N.$$

We have, from Lemma 2.3 and (2.6)

$$\begin{aligned} \|x - G_N(x)\|_{\mathbb{B}} &\leq \|x - p_N(x)\|_{\mathbb{B}} + \|p_N(x) - S_P(x)\|_{\mathbb{B}} + \|S_P(x) - S_{\Gamma}(x)\|_{\mathbb{B}} \\ &= \|x - p_N(x)\|_{\mathbb{B}} + \|S_P(x - p_N(x))\|_{\mathbb{B}} + \|S_P(x) - S_{\Gamma}(x)\|_{\mathbb{B}} \end{aligned}$$

$$\begin{aligned}
&\lesssim \left[1 + 8K^4 \left( \sum_{k=1}^{|\Gamma|} \mu(k) \frac{1}{k} \right) \right] \|x - p_N(x)\|_{\mathbb{B}} + \|S_{P \setminus \Gamma}(x) - S_{\Gamma \setminus P}(x)\|_{\mathbb{B}} \\
&\leq \left[1 + 8K^4 \left( \sum_{k=1}^{|\Gamma|} \mu(k) \frac{1}{k} \right) \right] (\sigma_N(x) + \epsilon) + \|S_{P \setminus \Gamma}(x)\|_{\mathbb{B}} + \|S_{\Gamma \setminus P}(x)\|_{\mathbb{B}}. \tag{2.7}
\end{aligned}$$

It is not difficult to find an upper bound for  $\|S_{\Gamma \setminus P}(x)\|_{\mathbb{B}}$ . Since  $p_N(x)$  is supported in  $P$  we have  $S_{\Gamma \setminus P}(p_N(x)) = 0$ . By Lemma 2.3

$$\begin{aligned}
\|S_{\Gamma \setminus P}(x)\|_{\mathbb{B}} &= \|S_{\Gamma \setminus P}(x - p_N(x))\|_{\mathbb{B}} \lesssim (8K^4) \left( \sum_{k=1}^{|\Gamma \setminus P|} \mu(k) \frac{1}{k} \right) \|x - p_N(x)\|_{\mathbb{B}} \\
&\leq (8K^4) \left( \sum_{k=1}^N \mu(k) \frac{1}{k} \right) [\sigma_N(x) + \epsilon]. \tag{2.8}
\end{aligned}$$

The bound for  $\|S_{P \setminus \Gamma}(x)\|_{\mathbb{B}}$  is more delicate. Use Lemma 2.1 with  $\eta(N) = h_r(N)$  to write

$$\|S_{P \setminus \Gamma}(x)\|_{\mathbb{B}} = \left\| \sum_{k \in P \setminus \Gamma} a_k(x) e_k \right\|_{\mathbb{B}} \leq (2K) \sum_{k=1}^{|P \setminus \Gamma|} a_k^*(S_{P \setminus \Gamma}(x)) h_r(k) \frac{1}{k}.$$

If  $k \in \Gamma \setminus P$  and  $s \in P \setminus \Gamma$  we have  $a_s^*(S_{P \setminus \Gamma}(x)) \leq a_k^*(S_{\Gamma \setminus P}(x))$  by construction of the thresholding greedy algorithm since  $\min_{\Gamma} |a_k(x)| \geq \max_{\mathbb{N} \setminus \Gamma} |a_k(x)|$ . Also, since  $|P| = N = |\Gamma|$  we have  $|P \setminus \Gamma| = |\Gamma \setminus P|$ . Thus

$$\|S_{P \setminus \Gamma}(x)\|_{\mathbb{B}} \leq (2K) \sum_{k=1}^{|\Gamma \setminus P|} a_k^*(S_{\Gamma \setminus P}(x)) h_r(k) \frac{1}{k}.$$

We use that  $S_{\Gamma \setminus P}(p_N(x)) = 0$  and Lemma 2.2 with  $\eta = h_l$  to obtain

$$\begin{aligned}
\|S_{P \setminus \Gamma}(x)\|_{\mathbb{B}} &\lesssim (2K) \sum_{k=1}^{|\Gamma \setminus P|} a_k^*(S_{\Gamma \setminus P}(x - p_N(x))) h_r(k) \frac{1}{k} \\
&= (2K) \sum_{k=1}^{|\Gamma \setminus P|} a_k^*(G_{|\Gamma \setminus P|}(x - p_N(x))) \frac{h_r(k)}{h_l(k)} h_l(k) \frac{1}{k} \\
&\leq (2K) [\sup_k a_k^*(G_{|\Gamma \setminus P|}(x - p_N(x))) h_l(k)] \left( \sum_{k=1}^N \mu(k) \frac{1}{k} \right) \\
&\leq (2K) (4K^3) \|G_{|\Gamma \setminus P|}(x - p_N(x))\|_{\mathbb{B}} \left( \sum_{k=1}^N \mu(k) \frac{1}{k} \right).
\end{aligned}$$

We use now that  $\mathcal{B}$  is a quasi-greedy basis to write

$$\begin{aligned}
\|S_{P \setminus \Gamma}(x)\|_{\mathbb{B}} &\lesssim (8K^5) \|x - p_N(x)\|_{\mathbb{B}} \left( \sum_{k=1}^N \mu(k) \frac{1}{k} \right) \\
&\leq (8K^5) (\sigma_N(x) + \epsilon) \left( \sum_{k=1}^N \mu(k) \frac{1}{k} \right). \tag{2.9}
\end{aligned}$$

Replacing (2.8) and (2.9) in (2.7), and letting  $\epsilon \rightarrow 0$  we obtain the result stated in Theorem 1.1.

### 3. COMMENTS AND QUESTIONS

**3.1.** Let  $\mathcal{B}$  be a seminormalized quasi-greedy basis in a real Hilbert space  $\mathbb{H}$ . Since  $\mathcal{B} = \{e_k\}_{k=1}^\infty$  is unconditional for constant coefficients (see Proposition 2 in [9]) it follows from Kintchine's inequality that

$$\left\| \sum_{k \in \Gamma} e_k \right\|_{\mathbb{H}} \approx \sqrt{|\Gamma|}.$$

Thus, in this case we can take  $\eta(N) = N^{1/2}$  in Lemma 2.1 and Lemma 2.2, giving us Theorem 3 from [9].

**3.2.** Let  $\mathcal{B} = \{e_k\}_{k=1}^\infty$  be a quasi-greedy basis in  $L^p(\mathbb{T}^d)$ . If  $2 \leq p < \infty$  the space  $L^p(\mathbb{T}^d)$  has type 2 and cotype  $p$ . Thus

$$C'_p |\Gamma|^{1/p} \leq \left\| \sum_{\Gamma} e_k \right\| \leq C_p |\Gamma|^{1/2}, \quad \Gamma \subset \mathbb{N}. \quad (3.1)$$

Taking  $\eta(N) \approx N^{1/2}$  in Lemma 2.1 and  $\eta(N) \approx N^{1/p}$  in Lemma 2.2 we obtain Theorem 11 from [7] for the case  $2 \leq p < \infty$ .

The case  $1 < p \leq 2$  of Theorem 11 from [7] is obtained by observing that for this range of  $p$ 's the space  $L^p(\mathbb{T}^d)$  has type  $p$  and cotype 2, so that

$$C'_p |\Gamma|^{1/2} \leq \left\| \sum_{\Gamma} e_k \right\| \leq C_p |\Gamma|^{1/p}, \quad \Gamma \subset \mathbb{N}. \quad (3.2)$$

**3.3.** The proofs of Lemmata 2.1 and 2.2 follow the pattern of the proofs of 1.  $\Rightarrow$  2. in Theorem 3.1 and Theorem 4.2 from [3] for the limiting case " $\alpha = 0$ ".

**3.4.** As in [9] write, for  $N = 1, 2, 3, \dots$

$$e_N(\mathbb{B}) = e_N := \sup_{x \in \mathbb{B}} \frac{\|x - G_n(x)\|_{\mathbb{B}}}{\sigma_N(x)}, \quad \left(\frac{0}{0} = 1\right).$$

Theorem 1.1 shows that for a quasi-greedy basis in a real Banach space

$$e_N \leq C \left( \sum_{k=1}^N \mu(k) \frac{1}{k} \right) \lesssim \mu(N) \log N, \quad N \in \mathbb{N}.$$

For unconditional bases, Theorem 4 from [9] shows that

$$e_N \approx \mu(N), \quad N \in \mathbb{N}. \quad (3.3)$$

The same argument that proves (3.3) can be used to prove the following result: for a quasi-greedy basis  $\mathcal{B}$  in a real Banach space  $\mathbb{B}$

$$\tilde{e}_N \approx \mu(N), \quad N \in \mathbb{N}. \quad (3.4)$$

were

$$\tilde{e}_N(\mathbb{B}) = \tilde{e}_N := \sup_{x \in \mathbb{B}} \frac{\|x - G_n(x)\|_{\mathbb{B}}}{\tilde{\sigma}_N(x)}, \quad \left(\frac{0}{0} = 1\right).$$

and

$$\tilde{\sigma}_N(x) = \tilde{\sigma}_N(x; \mathcal{B}, \mathbb{B}) := \inf \left\{ \|x - \sum_{k \in \Gamma} a_k(x) e_k\|_{\mathbb{B}} : |\Gamma| \leq N \right\}$$

is the *expansional* best approximation to  $x = \sum_{k \in \mathbb{N}} a_k(x) e_k \in \mathbb{B}$ .

Since  $\sigma_N(x) \leq \tilde{\sigma}_N(x)$ , for a quasi-greedy basis we have by (3.4) and Theorem 1.1

$$\mu(N) \lesssim \tilde{e}_N(\mathcal{B}) \leq e_N(\mathcal{B}) \lesssim \mu(N) \log N. \quad (3.5)$$

By the comments that follow the statement of Theorem 1.1 if  $\mu$  has positive dilation index,  $\mu(N) \lesssim \tilde{e}_N(\mathcal{B}) \leq e_N(\mathcal{B}) \lesssim \mu(N)$ . The last inequality in (3.5) was proved in [9] for the Hilbert space case (see the Remark that follows the proof of Theorem 5 in [9]).

**QUESTION 1.** Is the inequality on the right hand side of (3.5) sharp? That is, is it possible to find a quasi-greedy basis  $\mathcal{B}$  such that  $e_N(\mathcal{B}) \approx \mu(N) \log N$ ? This question appears in [7] for the Hilbert space case (see paragraph that follows Theorem 10 in [7]).

**QUESTION 2.** Is it true that for a quasi-greedy basis  $\tilde{\sigma}_N(x) \lesssim \sigma_N(x) \log N$ ? If the answer is "yes" then by (3.4) we will have  $e_N(\mathcal{B}) \lesssim \tilde{e}_N(\mathcal{B}) \log N \lesssim \mu(N) \log N$ , given another proof of the right hand side of (3.5).

**3.5.** For a quasi-greedy basis  $\mathcal{B} = \{e_k\}_{k=1}^{\infty}$  in  $L^p(\mathbb{T}^d)$ , inequalities 3.1 and 3.2 (or type and cotype properties of  $L^p(\mathbb{T}^d)$ ) show that  $\mu(N) \lesssim N^{|\frac{1}{p}-\frac{1}{2}|}$ . By the comments that follow the statement of Theorem 1.1, if  $p \neq 2$  and  $1 < p < \infty$ ,  $e_N(\mathbb{B}) \lesssim N^{|\frac{1}{p}-\frac{1}{2}|}$ , proving Theorem 1.1 from [8]. (Notice that  $w(N) := N^{|\frac{1}{p}-\frac{1}{2}|}$  has positive dilation index if  $p \neq 2$ .) For  $p = 2$  we have  $e_N(\mathcal{B}) \lesssim \log N$  by Theorem 1.1.

Consider now the trigonometric system  $\mathcal{T}^d = \{e^{ikx} : k \in \mathbb{Z}^d\}$  in  $L^p(\mathbb{T}^d)$ ,  $1 \leq p \leq \infty$  (here  $L^\infty(\mathbb{T}^d)$  is  $C(\mathbb{T}^d)$ , the set of continuous functions in  $\mathbb{T}^d$ ). It is proved in [5] (Theorem 2.1) that

$$e_N(\mathcal{T}^d, L^p(\mathbb{T}^d)) \lesssim N^{|\frac{1}{p}-\frac{1}{2}|}, \quad 1 \leq p \leq \infty.$$

**QUESTION 3.** (Asked by V. N. Temlyakov at the *Concentration week on greedy algorithms in Banach spaces and compressed sensing* held on July 18-22 at Texas A&M University.)

a) Characterize those systems  $\mathcal{B}$  in  $L^p(\mathbb{T}^d)$ ,  $1 \leq p \leq \infty$ , such that  $e_N(\mathcal{T}^d, L^p(\mathbb{T}^d)) \lesssim N^{|\frac{1}{p}-\frac{1}{2}|}$ ,  $N \in \mathbb{N}$ . Notice that if  $1 < p \neq 2 < \infty$ , the characterization must be satisfy by  $\mathcal{T}^d$  as well as any quasi-greedy basis.

More generally,

b) Let  $v(N)$  be an increasing function of  $N$ . Characterize those systems  $\mathcal{B}$  in a Banach space  $\mathbb{B}$  for which  $e_N(\mathcal{B}, \mathbb{B}) \lesssim v(N)$ .

#### 4. APPENDIX

**Lemma 4.1.** *If  $\mathcal{B}$  is a quasi-greedy basis in a Banach space  $\mathbb{B}$ , the function  $\mu$  defined in (1.3) is doubling.*

*Proof.* It is proved in [9] and [1] that for a quasi-greedy basis  $\mathbb{B} = \{e_k\}_{k=1}^\infty$  with quasi-greedy constant  $K$ , if  $B \subset A \subset \mathbb{N}$  (finite sets) then

$$\left\| \sum_{k \in B} e_k \right\|_{\mathbb{B}} \leq K \left\| \sum_{k \in A} e_k \right\|_{\mathbb{B}}. \quad (4.1)$$

We have to prove that  $\mu(2N) \leq D\mu(N)$  for some  $D$  independent of  $N$ . Since  $\mu(2N)$  is defined as a supremum over the finite set  $1 \leq k \leq 2N$ , there exists  $k_0 \leq 2N$  such that  $\mu(2N) = h_r(k_0)/h_l(k_0)$ . Notice that  $h_r$  is doubling with doubling constant 2 by the triangle inequality.

Suppose first that  $k_0 = 2s \leq 2N$  is even. From (4.1) we deduce  $h_l(s) \leq Kh_l(2s)$ . Hence

$$\mu(2N) = \frac{h_r(2s)}{h_l(2s)} \leq (2K) \frac{h_r(s)}{h_l(s)} \leq (2K)\mu(N)$$

since  $s \leq N$ .

Assume now that  $k_0 = 2s + 1$  is odd. Since  $2s + 1 = k_0 \leq 2N$  we deduce  $s \leq N - \frac{1}{2}$ , and since  $s$  is an integer  $s \leq N - 1$ . From (4.1) we deduce  $h_r(2s + 1) \leq Kh_r(2s + 2)$  and  $h_l(s + 1) \leq Kh_l(2s + 1)$ . Hence

$$\mu(2N) = \frac{h_r(2s + 1)}{h_l(2s + 1)} \leq K^2 \frac{h_r(2s + 2)}{h_l(s + 1)} \leq 2K^2 \frac{h_r(s + 1)}{h_l(s + 1)} \leq (2K^2)\mu(N)$$

since  $s + 1 \leq N$ . □

**Acknowledgements.** This work started when the author participated in the *Concentration week on greedy algorithms in Banach spaces and compressed sensing* held on July 18-22 at Texas A&M University. I would like to express my gratitude to the Organizing Committee for the invitation to participate in this meeting.

## REFERENCES

- [1] S.J. DILWORTH, N.J. KALTON, D. KUTZAROVA, V.N. TEMLYAKOV, *The Thresholding Greedy Algorithm, Greedy Bases, and Duality*, Constr. Approx., 19, (2003), 575–597.
- [2] G. GARRIGÓS, E. HERNÁNDEZ, J.M. MARTELL, *Wavelets, Orlicz spaces and greedy bases*, Appl. Compt. Harmon. Anal., (24): (2008), 70–93.
- [3] G. GARRIGÓS, E. HERNÁNDEZ, M. DE NATIVIDADE, *Democracy functions and optimal embeddings for approximation spaces*, Adv. in Comp. Math. 2011 (Accepted)
- [4] A. KAMONT, V.N. TEMLYAKOV, *Greedy approximation and the multivariate Haar system*, Studia Math, 161 (3), (2004), 199–223.
- [5] V. N. TEMLYAKOV, *Greedy algorithm and n- term trigonometric approximation*, Const.Approx., 14, (1998), 569–587.
- [6] V. N. TEMLYAKOV, *Greedy approximation*, Acta Numerica (2008), 235–409.
- [7] V. N. TEMLYAKOV, M. YANG, P. YE, *Greedy approximation with regard to non-greedy bases*, Adv. in Comp. Math., 34, (2011), 219–337.
- [8] V. N. TEMLYAKOV, M. YANG, P. YE, *Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases*, Preprint.
- [9] P. WOJSTASZCZYK, *Greedy Algorithm for General Biorthogonal Systems*, Journal of Approximation Theory, 107, (2000), 293–314.

EUGENIO, HERNÁNDEZ, DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049, MADRID, SPAIN

*E-mail address:* eugenio.hernandez@uam.es